Relativity of Entanglement

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It has recently been suggested that various entanglement measures for bipartite mixed states do not in general give the same ordering even in the asymptotic cases [S. Virmani and M. B. Plenio, Phys. Lett. A **268**, 31 (2000)]. That is, for two certain mixed states, the order of the degree of entanglement depends on the measures. Therefore, incomparable pairs of mixed states which cannot be transformed to each other with unit efficiency by any combinations of local quantum operations and classical communications exist. We make an analogy of the relativity of the order of the degree of entanglement to the relativity of temporal orders in the special theory of relativity.

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Quantum entanglement led to a controversy over the Einstein-Podolsky-Rosen experiment [1] and to the nonlocality of quantum mechanics [2]. On the other hand, entanglement is one of the key ingredients in quantum information processing [3]. For example, a speedup in quantum computation [4] is obtained through parallel quantum operations on massively superposed states, which are, in general, entangled.

Recently, it has been shown that different entanglement measures can give rise to different orderings for mixed states of bipartite systems [5,6]. In this paper, we make an analogy between the non-uniqueness (or relativity) of the order of the degree of entanglement and the relativity of temporal order in the special theory of relativity [7]. The temporal order of two certain events can be reversed, depending on the observers' reference frames when the two events are not causally conntected. This is analogous to the following fact: The order of the degree of entanglement for two certain mixed states can be reversed, depending on the entanglement measures. A trajectory of quantum states induced by a local quantum operation assisted by classical communications (LOCCs) corresponds to a dynamic trajectory of a particle in special relativity.

A few authors have proposed serveral entanglement measures, which quantify the degree of entanglement of quantum states, such as negativity of entanglement E_n [8–10], entanglement of cost E_C, entanglement of distillation E_D [11], and quantum relative entropy of entanglement E_r [12]. These are *reasonable* measures in the sense that they satisfy the three necessary conditions [11–13]: (C1) $E(\rho)=0$ if and only if the density operator ρ of a bipartite system is separable, (C2) local unitary operations leave $E(\rho)$ invariant, and (C3) the expected degree of entanglement does not increase under any LOCCs.

The entanglement of quantum states has a rich structure and require, in general, multiple parameters for complete characterization. It was shown that a single parameter is sufficient to characterize pure entangled states of bipartite systems in the asymptotic case [14,15]. In this special case, all entanglement measures are reduced to a unique measure [16,17]. In general, however, the entanglement requires multiple parameters for characterization, for example, $N-1$ parameters for pure entangled states of bipartite N-dimensional systems [15,18,19].

For better understanding and manipulation of entangled states, quantum states must be classified as well as possible. The degree of entanglement of an entangled pair can be compared by investigating the existence of LOCCs that transform one state to another. One state can be said to be more entangled than another if the former can be transformed to the latter by LOCCs with unit efficiency. However, some pairs of entangled states exist such that one cannot be transformed to the other by any LOCCs. These pairs of states cannot be compared with each other; in other words, they are 'incomparable' (A simple example is given in Ref. [20]).

Consideration of orderings in degree of entanglement are important in analyzing the structure of entanglement. Two certain entanglement measures E_A and E_B are defined to have the same ordering if they satisfy the following condition for any two density operators ρ_i and ρ_j;

$$E_A(\rho_i) > E_A(\rho_j) \Leftrightarrow E_B(\rho_i) > E_B(\rho_j). \tag{1}$$

Recently, Virmani and Plenio showed that if two certain entanglement measures, which are identical for pure states, give different degrees of entanglement for some mixed states, the orderings of the two measures are different for those mixed states [5]. For examples, the existence of a bound entangled state and some losses [21,22] in purification processes cause the entanglement of cost E_C to be strictly greater than entanglement of distillation E_D even in asymptotic cases. (In the finite case, it was explicitly shown by numerical analysis that the entanglement of formation E_f [11] and the negativity of entanglement E_n give different orderings [6].) Thus, the orderings in the degree of entanglement depend on the measures. This fact suggests that the various entanglement measures do not have to give the same ordering for mixed states even in asymptotic cases.

We note that, although the conclusion appears to be odd, it does not give rise to any bare contradictions. Let us consider two density operators ρ_i and ρ_j. The fact

that the order depends on the entanglement measures obviously means that degree of entanglement of ρ_i is less than that of ρ_j in one of the two measures and vice versa in the other. That is, we have either

$$E_A(\rho_1) > E_A(\rho_2) \text{ and } E_B(\rho_1) < E_B(\rho_2), \tag{2}$$

or

$$E_A(\rho_1) < E_A(\rho_2) \text{ and } E_B(\rho_1) > E_B(\rho_2). \tag{3}$$

A quantum state with density operator ρ_i cannot be transformed to one with ρ_j by any LOCCs due to condition (C3) if the degree of entanglement of ρ_i is less than ρ_j in any of the reasonable entanglement measures. (With less efficiency than one, forbidden paths may be allowed by LOCCs.) Thus, the pairs of states are incomparable [20], implying the necessity of multiple parameters for characterizing the entanglement of bipartite mixed states. Now, it is interesting to note that this fact is analogous to a fact of the special relativity. The temporal order of two certain events depends on observers' reference frames [7]. Although it appears to be odd, this fact does not give rise to contradictions because the two events cannot be causally connected (or can only be space-likely connected) in this case. Let us consider a map where each mixed state is coordinated by degree of entanglement of two certain measures, E_A and E_B. (See Fig. 1.) We can easily see that a point in the map cannot be moved by any local unitary operation due to condition (C2). Thus, a class of mixed states, which are equivalent within a local unitary operation, corresponds to a point in the map. In general, the converse is not true. The larger the number of reasonable entanglement measures we adopt, the more refined the coordination of density operators will become. If LOCCs are applied, a point can flow through a trajectory that always points in the lower-left direction from a point in the map. This fact is analogous to a fact in special relativity; each observer goes through a path in space-time that always points from a point to somewhere within the light cone. Here, a point and a trajectory in the map, respectively, correspond to an event and an observer's path in space-time. That there is no unique entanglement measure is also analogous to there being no preferred reference frame in special relativity. Multiplicity of measures does not mean oddity, but a higher dimensional structure of degree of entanglement of quantum states. In fact, the conclusion that multiple parameters are needed has been logically obtained by Vidal and by Bennett *et al.* in the case of finite pure states where incomparable states exist [15,19]. What we have shown here is that existence of incomparable states and multiple measures are in close analogy with some facts in the special theory of relativity.

In conclusion, various entanglement measures for quantum bipartite mixed states do not give the same ordering in general [5], even in asmyptotic cases. That is, two certain mixed states' ordering in the degree of entanglement depends on the entanglement measures. However, such pairs of mixed states cannot be transformed to each other by LOCCs, so they are incomparble. We make an analogy between these facts and the relativity of temporal order in the special theory of relativity. The non-uniqueness of the entanglement measure is also in analogy with the non-existence of a preferred reference frame. Our hope is that this analogy will inspire other ideas which may lead us to a better understanding of the structure of entanglement measures. It is notable that after this work the analogy was persued in the case of finite pure-state entanglement [18].

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Fig. 1; Consider a map where each mixed state is coordinated by the degrees of entanglement of two certain measures, E_A and E_B. If LOCCs are applied, a point can flow through a trajectory that always points in the lower-left direction from a point in the map. This fact is in analogy with special relativity where each observer goes through a path in space-time that always points to somewhere within the light-cone from a point. A point and a trajectory in the map, respectively, correspond to an event and an observer's path in space-time. *The states corresponding to p and q are incomparable since they cannot be transformed, with unit efficiency, to each other by any LOCCs.*